1

A P P E N D I X III

AN ALGORITHM FOR LEAST-SQUARES RATIONAL APPROXIMATION OF COMPLEX FUNCTIONS ON THE UNIT CIRCLE

Bo Jansson

Seismological Institute

University of Uppsala

Uppsala, Sweden

Contract No. AF 61(052)-702, Mod. 1-4

July 26, 1965

Prepared for

GEOPHYSICS RESEARCH DIRECTORATE

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES

OFFICE OF AEROSPACE RESEARCH

UNITED STATES AIR FORCE

BEDFORD, MASSACHUSETTS

WORK SPONSORED BY ADVANCED RESEARCH PROJECTS AGENCY

PROJECT VELA-UNIFORM

ARPA Order No. 292, Amendment No. 12, dated 20 Dec 1963

Project 8652, Task 865203

Project Code 4810, Program Element 62506015

8

BEST
AVAILABLE COPY

AN ALGORITHM FOR LEAST-SQUARES RATIONAL APPROXIMATION OF COMPLEX FUNCTIONS ON THE UNIT CIRCLE

Bo Jansson

Abstract. An algorithm for rational approximation on the unit circle will be given. An immediate application is computation of recursive filters for use on a digital computer.

1. Definitions

Let $f(\omega)$ be a given complex-valued function defined for real values of ω, and let

$$R_{m,n}(\omega) = \frac{a_0 + a_1 e^{-i\omega} + \dots + a_m e^{-im\omega}}{b_0 + b_1 e^{-i\omega} + \dots + b_n e^{-in\omega}} \tag{1}$$

be a complex-valued rational approximating function defined for real values of ω. The sets $A = \{a_0, a_1, \dots , a_m\}$ and $B = \{b_0, b_1, \dots , b_n\}$ are real parameters of $R_{m,n}(\omega)$. In the formulas below a_i and b_i are assumed to be equal to zero if they do not belong to A or B, i.e. if their subscripts are outside the intervals 0 - m or 0 - n respectively. We will always assume that $m \geq n$.

2. Least-squares approximation

We want to determine the parameters $a_0, a_1, \dots , a_m$ and $b_0, b_1, \dots , b_n$ such that

$$L_2(f - R) = \left[\int_0^{2\pi} \left| f(\omega) - R_{m,n}(\omega) \right|^2 d\omega \right]^{1/2} \tag{2}$$

attains a minimum (assuming it exists). The "linear" approach of taking the partial derivatives with respect to all the parameters is not feasible, as it will result in a system of non-linear equations which is hard to solve. We will avoid this difficulty by using an algorithm given by the author (Jansson, 1965).

3. The algorithm

Following the algorithm mentioned in the last paragraph we start by adding a real constant h to $f(\omega)$, thus having $g(\omega) = f(\omega) + h$. Let $B^0 = \{1, 0, \dots, 0, \dots, 0\}$ and consider $L_2(g - R)$ as a function of $a_0, a_1, \dots, a_m$ only. We can minimize $L_2(g - R)$ by minimizing $L_2(g - R)^2$, and this makes the analysis simpler. It may be shown that a unique minimum exists. Furthermore, it is clear that $L_2(g - R)$ does not have a maximum value, for we can place $R(\omega)$ as far away from $f(x)$ as we please. Since $L_2(g - R)^2$ is a differentiable function of $a_0, a_1, \dots, a_m$, the minimum must occur where all of the partial derivatives

$$\frac{\partial L_2(g - R)^2}{\partial a_k}, \qquad k = 1, 2, \dots, m, \tag{3}$$

are zero.

From the fact that $L_2(g - R)$ is a strictly convex function of $a_0, a_1, \dots, a_m$, it follows that there is only one set of parameters for which all of the partial derivatives are zero.

We have

$$I = |g(\omega) - R_{m,n}(\omega)|^2 =$$

$$= |g(\omega)|^2 + |R_{m,n}(\omega)|^2 - \mathrm{Re}\{g(\omega)\, R_{m,n}(\omega)\}, \tag{4}$$

where

$$\left|R_{m,n}(\omega)\right|^2 =$$

$$= \frac{(a_0^2 + a_1^2 + \ldots + a_n^2) + 2(a_0\, a_1 + a_1\, a_2 + \ldots + a_{n-1}\, a_m) \cos\omega + \ldots + 2(a_0\, a_m) \cos m\,\omega}{(b_0^2 + b_1^2 + \ldots + b_n^2) + 2(b_0\, b_1 + b_1\, b_2 + \ldots + b_{n-1}\, b_n) \cos\omega + \ldots + 2(b_0\, b_n) \cos n\,\omega}$$

and

$$\mathrm{Re}\{g(\omega)\, R_{m,n}(\omega)\} =$$

$$= \Big[\mathrm{Re}\{g(\omega)\}\, \mathrm{Re}\{P_m(\omega)\, \overline{Q_n(\omega)}\} -$$

$$- \mathrm{Im}\ \{g(\omega)\}\ \mathrm{Im}\ \{P_m(\omega)\, \overline{Q(\omega)}\}\Big] / |Q_n(\omega)|^2.$$

We have that

$$P_m(\omega)\, \overline{Q_n(\omega)} = a_0\, b_n\, e^{in\omega} + a_0\, b_{n-1}\, e^{i(n-1)\omega} + \ldots +$$

$$+ (a_0\, b_0 + a_1\, b_1 + \ldots + a_n b_n) + (a_1\, b_0 + a_2\, b_1 + \ldots + a_{n+1}\, b_n)\, e^{-i\omega} +$$

$$+ \ldots + a_m\, b_0\, e^{-im\omega}. \tag{5}$$

It is now easy to compute the partial derivatives

$$\frac{\partial I}{\partial a_k} = 2\Big[\sum_{j=0}^{m} (a_{k+j} + a_{k-j}) \cos j\,\omega -$$

$$- \mathrm{Re}\{g(\omega)\} \sum_{j=-m}^{m} b_{k+j} \cos j\,\omega + \mathrm{Im}\ \{g(\omega)\} \sum_{j=-m}^{m} b_{k+j} \sin j\,\omega\Big] /$$

$$/ |Q_n(\omega)|^2 \tag{6}$$

Setting all the partial derivatives equal to zero and substituting into (3) we get the matrix equation

$$M_A \cdot A^0 = G_A \tag{7}$$

M_A is a m x m Toeplitz matrix, A^0 is the column vector $(a_0, a_1, \dots, a_m)^T$, and G_{Ai} a column vector. Let us write out the elements of M_A and G_A

$$M_{Aij} = \int_0^{2\pi} \frac{\cos |i - j| \omega}{|Q_n(\omega)|^2} d\,\omega$$

$$G_{Ai} = \int_0^{2\pi} \left[\mathrm{Re}\{g(\omega)\} \sum_{j=-m}^{m} b_{j+i} \cos j\,\omega - \right.$$

$$\left. - \mathrm{Im}\ \{g(\omega)\} \sum_{j=-m}^{m} b_{j+i} \sin j\,\omega \right] \cdot |Q_n(\omega)|^{-2} d\,\omega \tag{8}$$

To solve for A we just have to compute

$$A^0 = M_A^{-1}\, G_A$$

which is very easy since M is a Toeplitz matrix that can be inverted in cm^2 operations. Here c is a constant and m is the number of rows and columns. An algorithm for this purpose has been given by Robinson (1963).

We have now done the first step and computed a trial solution A^0. Let us in the next step compute a trial solution B^1. We will do this by minimizing

$$L_2\left(g^2\left(\frac{1}{R} - \frac{1}{g}\right)\right) = \left[\int_0^{2\pi} \left|g^2(\omega)\left(\frac{1}{R_{m,n}(\omega)} - \frac{1}{g(\omega)}\right)\right|^2 d\,\omega\right]^{1/2} \tag{9}$$

Let us write

$$I = \left|g^2(\omega)\left(\frac{1}{R_{m,n}(\omega)} - \frac{1}{g(\omega)}\right)\right|^2 =$$

$$= \left|g^2(\omega)\left(\frac{Q_n(\omega)}{P_m(\omega)} - \frac{1}{g(\omega)}\right)\right|^2 =$$

$$= \left|g^2(\omega)\,\frac{Q_n(\omega)}{P_m(\omega)}\right|^2 - |g(\omega)|^2 - 2\mathrm{Re}\left\{g^2(\omega)\,\frac{Q_n(\omega)}{P_m(\omega)} - g(\omega)\right\} \tag{10}$$

The parameters B enter linearly in the expression above so that we may take the partial derivatives and solve for B^1 just as we solved for A^0.

Again we get a matrix equation

$$M_B\, B^1 = G_B \tag{11}$$

where M_B is a Toeplitz matrix and G_B is a column vector. We solve for B^1 by computing

$$B^1 = M_B^{-1}\, G_B.$$

The elements of M_B and G_B may be written

$$M_{Bij} = \int_0^{2\pi} \frac{g^2(\omega)\cos|i-j|\,\omega}{|P_m(\omega)|^2}\, d\,\omega$$

$$G_{Bij} = \int_0^{\pi} \left[\operatorname{Re}\{g^2(\omega)\} \sum_{j=-m}^{m} a_{j+i} \cos j\,\omega - \right.$$

$$\left. - \operatorname{Im}\{g^2(\omega)\} \sum_{j=-m}^{m} a_{j+i} \sin j\,\omega \right] \cdot |P_m(\omega)|^{-2}\, d\,\omega \qquad (12)$$

We have now completed one iteration of the algorithm and we just have to continue to determine A^s and B^s, A^{s+1} and B^{s+1} etc until we are close enough to the point in the parameter space AUB where the minimum (if it exists) is located. We have then computed a sufficiently good approximation $R^g_{m,n}(\omega)$ to $g(\omega)$. Obviously

$$R^f_{m,n}(\omega) = R^g_{m,n}(\omega) - h = \frac{P^g_m(\omega) - h\,Q^g_n(\omega)}{Q^g_n(\omega)} = \frac{P^f_m(\omega)}{Q^g_n(\omega)} \qquad (13)$$

is an equally good approximation to $f(\omega)$.

References

Jansson, Bo, 1965, An Algorithm for Approximation by Rational Functions, AF 61(052)-702, No. 23.

Rice, John, 1964, The Approximation of Functions, Addison-Wesley.

Robinson, Enders, 1963, Mathematical Development of Discrete Filters for the Detection of Nuclear Explosions, Journal of Geophysical Research, 68:5559-5567.

Acknowledgment

This research was carried out at the Seismological Institute, Uppsala, with the sponsorship of the Cambridge Research Laboratories of the Office of Aerospace Research, United States Air Force, through its European Office, as part of the Advanced Research Projects Agency's project Vela-Uniform, under contract AF 61(052)-702.